

Spherical Schrödinger Hamiltonians: Spectral Analysis and Time Decay

Luca Fanelli

Abstract In this survey, we review recent results concerning the canonical dispersive flow e^{itH} led by a Schrödinger Hamiltonian H . We study, in particular, how the time decay of space L^p -norms depends on the frequency localization of the initial datum with respect to the some suitable spherical expansion. A quite complete description of the phenomenon is given in terms of the eigenvalues and eigenfunctions of the restriction of H to the unit sphere, and a comparison with some uncertainty inequality is presented.

1 Introduction

For $\psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$, let us consider the free Schrödinger equation

$$\partial_t \psi = i\Delta \psi, \quad \psi(0, x) = \psi_0(x). \quad (1)$$

Solving (1) with initial datum $\psi_0(x) \in L^2(\mathbb{R}^d)$ is to find a wavefunction $\psi \in \mathcal{C}^1(\mathbb{R}; L^2(\mathbb{R}^d))$ such that $\widehat{\psi}(t, \xi) = e^{-it|\xi|^2} \widehat{\psi}_0(\xi)$, the hat denoting the Fourier transform in the x -variable

$$\widehat{\psi}(t, \xi) := \int_{\mathbb{R}^d} e^{-it\xi} \psi(x) dx.$$

Computing the distributional Fourier transform of $e^{-it|\xi|^2}$, one finds that the unique solution to (1), in the above sense, is given by

$$\psi(t, x) = (4\pi it)^{-\frac{d}{2}} e^{-i\frac{|x|^2}{4t}} * \psi_0(x) = (4\pi it)^{-\frac{d}{2}} e^{-i\frac{|x|^2}{4t}} \int_{\mathbb{R}^d} e^{i\frac{x \cdot y}{2t}} e^{-i\frac{|y|^2}{4t}} \psi_0(y) dy. \quad (2)$$

Luca Fanelli

Name, Dipartimento di Matematica, SAPIENZA Università di Roma, P. le Aldo Moro 5, 00185 Roma, e-mail: fanelli@mat.uniroma1.it

From now on, we will denote by $e^{it\Delta}$ the one-parameter flow on $L^2(\mathbb{R}^d)$ defined by formula (2), namely $e^{it\Delta}\psi_0(\cdot) = \psi(t, \cdot)$, being ψ as in (2). By Plancherel Theorem it follows that $e^{it\Delta}$ is unitary on $L^2(\mathbb{R}^d)$, namely

$$\left\| e^{it\Delta}\psi_0(\cdot) \right\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}, \quad \forall t \in \mathbb{R}. \quad (3)$$

By (2), it also immediately follows that

$$\left\| e^{it\Delta}\psi_0(\cdot) \right\|_{L^\infty(\mathbb{R}^d)} \leq C|t|^{-\frac{d}{2}} \|\psi_0\|_{L^1(\mathbb{R}^d)}, \quad \forall t \in \mathbb{R}, \quad (4)$$

with a constant $C > 0$ independent on t and ψ_0 . The last inequality, together with (3), gives by Riesz-Thorin the full list of *time decay estimates* for the free Schrödinger equation

$$\left\| e^{it\Delta}\psi_0(\cdot) \right\|_{L^p(\mathbb{R}^d)} \leq C|t|^{-d(\frac{1}{2}-\frac{1}{p})} \|\psi_0\|_{L^{p'}(\mathbb{R}^d)}, \quad \forall t \in \mathbb{R}, \quad \forall p \geq 2 \quad (5)$$

where the constant C only depends on p and d . Inequalities (5) turn out to be a crucial tool in Scattering Theory and Nonlinear Analysis; in particular, a suitable time average of the same leads to the so called *Strichartz estimates* (see the standard reference [24]), which play a fundamental role both for fixed point results and as Restriction Theorems for the Fourier transform:

$$\left\| e^{it\Delta}\psi_0 \right\|_{L_t^q L_x^r} \leq C \|\psi_0\|_{L^2(\mathbb{R}^d)}, \quad (6)$$

with $2/q = d/2 - d/r$, $q \geq 2$ and $(q, r, d) \neq (2, \infty, 2)$, and

$$\left\| e^{it\Delta}\psi_0(\cdot) \right\|_{L^p(\mathbb{R}^d)} := \left\| \left\| e^{it\Delta}\psi_0(\cdot) \right\|_{L^r(\mathbb{R}^d)} \right\|_{L^q(\mathbb{R})}.$$

From now on, we point our attention on estimate (4) and try to give it a deeper insight. First of all, it is clear by (2) that a crucial role is played by the plane wave $K(x, y) := e^{i\frac{xy}{2t}}$ which is uniformly bounded with respect to the x, y variables, for any fixed time $t \neq 0$, i.e.

$$\sup_{x, y \in \mathbb{R}^d} \left| e^{i\frac{xy}{2t}} \right| = 1 < \infty, \quad \forall t \neq 0. \quad (7)$$

We stress that a completely analogous behavior occurs when one solves, for positive times, the Heat Equation

$$\partial_t u = \Delta u, \quad u(0, x) = u_0(x) \in L^p(\mathbb{R}^d), \quad (8)$$

since the solution is given by the convolution

$$u(t, x) = (4\pi t)^{-\frac{d}{2}} e^{\frac{-|x|^2}{4t}} * u_0(x), \quad (t > 0) \quad (9)$$

for all $p \in [1, +\infty]$. This shows that (8) satisfies the same a priori estimates (5) as equation (1). Notice that (1) and (8) enjoy the same scaling invariance

$$f_\lambda(t, x) := f\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \quad \lambda > 0.$$

In addition, the Gaussian decay in (9) is much smoother than the oscillating character of the fundamental solution in (2), and leads to much stronger phenomena than the ones led by the dispersive flow $e^{it\Delta}$. Nevertheless, from the point of view of estimate (4) the behavior is the same for the flows $e^{t\Delta}, e^{it\Delta}$, when $t > 0$. Our first question is the following:

A *is the time decay of the flows $e^{t\Delta}, e^{it\Delta}$ related to the lowest frequency behavior of the corresponding fundamental solutions?*

We now pass to a more precise analysis of the decay estimate in (4), to describe some additional phenomenon which is hidden in formula (2). To this aim, let us recall the *Jacobi-Anger* expansion or plane waves, which combined with the Addition Theorem for spherical harmonics (see for example [22, formula (4.8.3), p. 116] and [3, Corollary 1]) yields

$$e^{ix \cdot y} = (2\pi)^{d/2} (|x||y|)^{-\frac{d-2}{2}} \sum_{\ell=0}^{\infty} i^\ell J_{\ell+\frac{d-2}{2}}(|x||y|) \left(\sum_{m=1}^{m_\ell} Y_{\ell,m}\left(\frac{x}{|x|}\right) \overline{Y_{\ell,m}\left(\frac{y}{|y|}\right)} \right) \quad (10)$$

for all $x, y \in \mathbb{R}^d$. Here J_ν denotes the ν -th Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}$$

and the $Y_{\ell,m}$ are usual spherical harmonics. Recalling that $J_\nu(t) \sim t^\nu$, for $\nu \geq 0$, as t goes to 0, we see that an additional time-decay, for t large is hidden in formula (2), in the term $e^{i\frac{x \cdot y}{t}}$. Roughly speaking, we expect that initial data which are localized higher frequencies (with respect to the spherical harmonics expansion) decay polynomially faster along a Schrödinger evolution, in suitable topologies. This leads to our second question:

B *how can the above described phenomenon be quantified, and how stable is it under lower-order perturbations?*

Looking to identity (10), the presence of spherical harmonics and special functions gives the hint that the spherical laplacian is playing an important role in the description of the above mentioned phenomena. The aim of this survey is to describe this role, giving partial answers to the above questions and leaving some open problems, corroborated by some recent results.

2 A stationary viewpoint: Hardy's Inequality

We devote a preliminary section to introduce an interesting stationary viewpoint of the above picture, related to some uncertainty inequalities. To this aim, we recall the well known *Hardy's inequality*:

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 dx, \quad (d \geq 3) \quad (11)$$

which holds for any function ψ such that $|\nabla \psi| \in L^2$. The constant in front of inequality (11) is sharp, and it is not attained on any function ψ for which the right-hand side is finite, as we see in a while. Inequality (11) can be rewritten in operator terms as

$$-\Delta - \frac{\lambda}{|x|^2} \geq 0, \quad \forall \lambda \leq \frac{(d-2)^2}{4} \quad (d \geq 3). \quad (12)$$

This has to be interpreted in the sense of the associated quadratic form. The proof of (11) relies on the following fact: given a symmetric operator \mathcal{S} and a skew-symmetric operator \mathcal{A} on L^2 , one can (formally) compute

$$0 \leq \int_{\mathbb{R}^d} |(\mathcal{A} + \mathcal{S})\psi|^2 dx = \int_{\mathbb{R}^d} |\mathcal{A}\psi|^2 dx + \int_{\mathbb{R}^d} |\mathcal{S}\psi|^2 dx - \int_{\mathbb{R}^d} \overline{\psi} [\mathcal{A}, \mathcal{S}] \psi dx,$$

where $[\mathcal{A}, \mathcal{S}] = \mathcal{A}\mathcal{S} - \mathcal{S}\mathcal{A}$. Then the choices

$$\mathcal{A} := \nabla, \quad \mathcal{S} := \frac{d-2}{2} \frac{x}{|x|^2} \quad \Rightarrow \quad [\mathcal{A}, \mathcal{S}] = \frac{(d-2)^2}{2|x|^2}$$

immediately give (11) for functions ψ smooth enough, and a regularization argument completes the proof. Also notice the equality in (11) is attained when $(\mathcal{A} + \mathcal{S})\psi \equiv 0$, which yields the maximizing function $\psi(x) = |x|^{1-\frac{d}{2}}$, and we see that $|\nabla \psi| \notin L^2$, as mentioned above. In addition, one immediately realizes that, given $\widetilde{\mathcal{A}} = \nabla_r = \nabla \cdot \frac{x}{|x|}$, then

$$[\widetilde{\mathcal{A}}, \mathcal{S}] = [\mathcal{A}, \mathcal{S}] = \frac{(d-2)^2}{2|x|^2},$$

which yields the more precise inequality

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\partial_r \psi(x)|^2 dx, \quad (d \geq 3) \quad (13)$$

In other words, inequality (13) shows that the angular component of $-\Delta$ is not playing a role in (11)-(12). To understand this fact, it is convenient to use spherical coordinates and write

$$\Delta = \partial_r^2 + \frac{d-1}{r} \partial_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{d-1}}, \quad (14)$$

being $\Delta_{\mathbb{S}^{d-1}}$ the spherical laplacian, i.e. the Laplace-Beltrami operator on the $(d-1)$ -dimensional unit sphere. We recall that $-\Delta_{\mathbb{S}^{d-1}}$ is a (positive) operator with compact inverse, hence it has purely point spectrum which accumulates at infinity, which is explicitly given by the set

$$\sigma(-\Delta_{\mathbb{S}^{d-1}}) = \sigma_p(-\Delta_{\mathbb{S}^{d-1}}) = \{\ell(\ell+d-2)\}_{\ell=0,1,2,\dots}. \quad (15)$$

Spherical harmonics $\{Y_{\ell,m}\}$ are associated eigenfunctions, which form a complete orthonormal set in $L^2(\mathbb{S}^{d-1})$. Denoting by H_ℓ the eigenspace associated to the ℓ -th eigenvalue of $-\Delta_{\mathbb{S}^{d-1}}$, by D_ℓ its algebraic dimension, and by $H_{\ell,m}$ the space generated by $Y_{\ell,m}$, we have the well known decomposition

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{\substack{\ell \geq 0 \\ 1 \leq m \leq D_\ell}} H_{\ell,m}$$

Therefore any function $\psi \in L^2(\mathbb{R}^d)$ has a (unique) expansion

$$\psi(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{D_\ell} \psi_{\ell,m}(r) Y_{\ell,m}(\omega) \quad x = r\omega, \quad r := |x| \quad (16)$$

and moreover

$$\|f(r\omega)\|_{L^2(\mathbb{S}^{d-1})}^2 = \sum_{\substack{\ell \geq 0 \\ 1 \leq m \leq D_\ell}} |f_{\ell,m}|^2.$$

We can hence use (14) to write

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \psi|^2 dx &= - \int_{\mathbb{R}^d} \bar{\psi} \Delta \psi dx \\ &= \underbrace{- \int_{\mathbb{R}^d} \bar{\psi} \left(\partial_r^2 \psi + \frac{d-1}{r} \partial_r \psi \right) dx}_{=: I} + \underbrace{\int_{\mathbb{R}^d} \frac{1}{|x|^2} \langle \psi, -\Delta_{\mathbb{S}^{d-1}} \psi \rangle_{L^2(\mathbb{S}^{d-1})} dx}_{=: II}. \end{aligned} \quad (17)$$

where the brackets $\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^{d-1})}$ denote the inner product in $L^2(\mathbb{S}^{d-1})$. Arguing as above we see that

$$I \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx, \quad (d \geq 3)$$

which is inequality (13). On the other hand, it follows by (15) that

$$II \geq 0,$$

therefore no additional contribution to (11) is given by $-\Delta_{\mathbb{S}^{d-1}}$. Nevertheless, given $\psi \in L^2(\mathbb{R}^{d-1})$, if $\psi_{0,1} = 0$ in the expansion (16) (notice that $H_{0,1}$ coincides with the space of L^2 -radial functions), then by (15) it follows that

$$\langle \psi, -\Delta_{\mathbb{S}^{d-1}} \psi \rangle_{L^2(\mathbb{S}^{d-1})} \geq (d-1) \|\psi(\omega)\|_{L^2(\mathbb{S}^{d-1})} \quad \text{if } \psi_{0,1} = 0$$

and inequality (13) improves:

$$\int_{\mathbb{R}^d} |\partial_r \psi(x)|^2 dx \geq \left(\frac{(d-2)^2}{4} + (d-1) \right) \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx, \quad (d \geq 2) \quad \psi_{0,1} = 0. \quad (18)$$

Notice that the previous gives a non trivial 2D-inequality, holding on functions ψ which are orthogonal to L^2 -radial functions. More in general, given $\psi \in L^2(\mathbb{R}^d)$, let

$$\ell_0 := \min\{\ell \in \mathbb{N} \text{ such that } \exists m = 1, \dots, D_\ell : \psi_{\ell,m} \neq 0\}.$$

Then, by (17), the following Hardy's inequality holds:

$$\int_{\mathbb{R}^d} |\partial_r \psi(x)|^2 dx \geq \left(\frac{(d-2)^2}{4} + \ell_0(\ell_0 + d - 2) \right) \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx. \quad (d \geq 1) \quad (19)$$

Inequality (19) is a quantitative stationary manifestation of the phenomenon described by question **B** in the Introduction. Here it is clear that the improvement comes from the angular component of the free Hamiltonian. In addition, the above arguments clearly suggest that the sharp constant in front of inequality (19) only depends the *lowest energies*, which is reminiscent of question **A** in the Introduction.

Having this in mind, we now see how linear lower-order perturbations of the free spherical Hamiltonian can perturb the spectral picture in (15), with consequences on the Hardy's inequality (19).

Example 1 (0-order perturbations). For $a \in \mathbb{R}$, consider the shifted Hamiltonians in dimension $d \geq 3$

$$H = -\Delta + \frac{a}{|x|^2}, \quad L = -\Delta_{\mathbb{S}^{d-1}} + a.$$

Clearly L only has point spectrum, which is just a shift of (15)

$$\sigma(L) = \sigma_p(L) = \{\ell(\ell + d - 2) + a\}_{\ell=0,1,2,\dots}$$

and spherical harmonics are still eigenfunctions. The corresponding Hardy's inequality is trivially

$$\left(\frac{(d-2)^2}{4} + a \right) \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 dx + a \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx. \quad (d \geq 3) \quad (20)$$

More in general, if $a = a(\omega) : \mathbb{S}^{d-1} \rightarrow \mathbb{R}$, then it is still true that L as only point spectrum, but the picture is more complicated. A typical phenomenon is the formation of clusters of eigenvalues around the (shifted) free eigenvalues. The size of the clusters depends on some universal dimensional quantity related to $a(\omega)$ (see e.g. the standard references [4, 21, 31, 32, 35] and Lemma 1 below). Moreover, for the lowest eigenvalue of L we have

$$\mu_0 := \min \sigma(L) = \inf_{\omega \in \mathbb{S}^{d-1}} \alpha(\omega).$$

One easily see by the same arguments as above that the following Hardy's inequality holds

$$\left(\frac{(d-2)^2}{4} + \mu_0 \right) \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} \bar{\psi} H \psi dx. \quad (21)$$

Example 2 (1st-order perturbations). Let $A \in L_{\text{loc}}^2(\mathbb{R}^d)$, and recall the *diamagnetic inequality*

$$|(-i\nabla + A)\psi(x)| \geq |\nabla|\psi(x)||.$$

This gives for free, together with (11), that

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|\psi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |(-i\nabla + A)\psi(x)|^2 dx, \quad (d \geq 3). \quad (22)$$

We wonder if an improvement to the best constant of inequality (22) can occur, due to the presence of an angular perturbation of the associated Hamiltonian, in the same style as in the above example. The main example we have in mind is given by the 2D-Aharonov-Bohm vector potential: for $\lambda \in \mathbb{R}$, consider let us denote by

$$A : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad A(x, y) := \lambda \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right)$$

and consider the following quadratic form

$$q[\psi] := \int_{\mathbb{R}^2} |(-i\nabla + A)\psi|^2 dx.$$

Since q is positive, we can consider the *Friedrichs' extension* of the self-adjoint Hamiltonian $H := -\nabla_A^2$, on the natural form domain induced by q (see Section 3 below for details). The angular component of H is the operator

$$L := (-i\nabla_{\mathbb{S}^1} + \mathcal{A}(\omega))^2, \quad \mathcal{A} : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad \mathcal{A}(x, y) = \lambda \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$

As above, L has compact inverse and its spectrum is explicitly given by

$$\sigma(L) = \sigma_p(L) = \{(\lambda - z)^2\}_{z \in \mathbb{Z}}.$$

Therefore, the lowest eigenvalue is given by

$$\mu_0 := \min \sigma(L) = \text{dist}(\lambda, \mathbb{Z})^2 \geq 0$$

and we gain the following 2D-Hardy's inequality, proved in [25]

$$\mu_0 \int_{\mathbb{R}^2} \frac{|\psi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^2} |(-i\nabla + A)\psi|^2 dx. \quad (23)$$

As soon as $\lambda \notin \mathbb{Z}$, this is an improvement with respect to the free case $A \equiv 0$, in which such an inequality cannot hold for any function ψ such that $|\nabla \psi| \in L^2(\mathbb{R}^2)$ (since the weight $|x|^{-2}$ is not locally integrable in 2D).

In view of the above considerations, we will restrict our attention, from now on, to some scaling-critical electromagnetic Hamiltonians and we will present some recent results which partially answer to questions **A** and **B** in the Introduction of this survey.

3 Decay estimates: main results

From now on, for any $x \in \mathbb{R}^d$, we denote by $x = r\omega$, $r = |x|$. Let

$$\mathbf{A} = \mathbf{A}(\omega) : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d, \quad a = a(\omega) : \mathbb{S}^{d-1} \rightarrow \mathbb{R}$$

to 0-degree homogenous functions, and consider the quadratic form

$$q[\psi] := \int_{\mathbb{R}^d} \left| \left(-i\nabla + \frac{\mathbf{A}(\omega)}{r} \right) \psi(x) \right|^2 dx + \int_{\mathbb{R}^d} \frac{a(\omega)}{r^2} |\psi(x)|^2 dx. \quad (24)$$

As we see in the sequel, under suitable conditions, a self-adjoint Hamiltonian

$$H := \left(-i\nabla + \frac{\mathbf{A}(\omega)}{r} \right)^2 + \frac{a(\omega)}{r^2}, \quad (25)$$

associated to q (Friedrichs' Extension) is well defined on a domain containing $L^2(\mathbb{R}^d)$, therefore the L^2 -initial value problem

$$\begin{cases} i\partial_t \psi = -iH\psi, \\ \psi(0) = \psi_0 \in L^2(\mathbb{R}^d), \end{cases} \quad (26)$$

for the wavefunction $\psi = \psi(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ makes sense. Here $d \geq 2$, and we choose a *transversal gauge* for the magnetic vector potential, i.e. we assume

$$\mathbf{A}(\omega) \cdot \omega = 0 \quad \text{for all } \omega \in \mathbb{S}^{d-1}. \quad (27)$$

Notice that equation (26) is invariant under the scaling $u_\lambda(x, t) := u(x/\lambda, t/\lambda^2)$, which is the same of the free Schrödinger equation.

The aim is to understand the role of the spherical operator L associated to H , defined by

$$L = \left(-i\nabla_{\mathbb{S}^{d-1}} + \mathbf{A} \right)^2 + a(\omega), \quad (28)$$

where $\nabla_{\mathbb{S}^{d-1}}$ is the spherical gradient on the unit sphere \mathbb{S}^{d-1} . The spectrum of the operator L is formed by a diverging sequence of real eigenvalues with finite multiplicity $\mu_0(\mathbf{A}, a) \leq \mu_1(\mathbf{A}, a) \leq \dots \leq \mu_k(\mathbf{A}, a) \leq \dots$ (see e.g. [17, Lemma A.5]), where

each eigenvalue is repeated according to its multiplicity. Moreover we have that $\lim_{k \rightarrow \infty} \mu_k(\mathbf{A}, a) = +\infty$. To each $k \geq 1$, we can associate a $L^2(\mathbb{S}^{d-1}, \mathbb{C})$ -normalized eigenfunction φ_k of the operator L on \mathbb{S}^{d-1} corresponding to the k -th eigenvalue $\mu_k(\mathbf{A}, a)$, i.e. satisfying

$$\begin{cases} L\varphi_k = \mu_k(\mathbf{A}, a) \varphi_k, & \text{in } \mathbb{S}^{d-1}, \\ \int_{\mathbb{S}^{d-1}} |\varphi_k|^2 dS(\theta) = 1. \end{cases} \quad (29)$$

In particular, if $d = 2$, φ_k are one-variable 2π -periodic functions, i.e. $\varphi_k(0) = \varphi_k(2\pi)$. Since the eigenvalues $\mu_k(\mathbf{A}, a)$ are repeated according to their multiplicity, exactly one eigenfunction φ_k corresponds to each index $k \geq 1$. We can choose the functions φ_k in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{d-1}, \mathbb{C})$. We also introduce the numbers

$$\alpha_k := \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}, \quad \beta_k := \sqrt{\left(\frac{d-2}{2}\right)^2 + \mu_k(\mathbf{A}, a)}, \quad (30)$$

so that $\beta_k = \frac{d-2}{2} - \alpha_k$, for $k = 1, 2, \dots$

Under the condition

$$\mu_0(\mathbf{A}, a) > -\frac{(d-2)^2}{4} \quad (31)$$

the quadratic form q in (24) associated to H is positive definite, and the Friedrichs' extension of H is well defined, with domain

$$\mathcal{D} := \left\{ f \in H_*^1(\mathbb{R}^d) : Hf \in L^2(\mathbb{R}^d) \right\}, \quad (32)$$

where $H_*^1(\mathbb{R}^d)$ is the completion of $C_c^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{C})$ with respect to the norm

$$\|f\|_{H_*^1(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left(|\nabla f(x)|^2 + \frac{|f(x)|^2}{|x|^2} + |f(x)|^2 \right) dx \right)^{1/2}.$$

By the Hardy's inequality (11), $H_*^1(\mathbb{R}^d) = H^1(\mathbb{R}^d)$ with equivalent norms if $d \geq 3$, while $H_*^1(\mathbb{R}^d)$ is strictly smaller than $H^1(\mathbb{R}^d)$ if $d = 2$. Furthermore, from condition (31) and [17, Lemma 2.2], it follows that $H_*^1(\mathbb{R}^d)$ coincides with the space obtained by completion of $C_c^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{C})$ with respect to the norm naturally associated to H , i.e.

$$q[\psi] + \|\psi\|_2^2.$$

We remark that H could be not essentially self-adjoint. Indeed, in the case $\mathbf{A} \equiv 0$, Kalf, Schmincke, Walter, and Wüst [23] and Simon [30] proved that H is essentially self-adjoint if and only if $\mu_0(\mathbf{0}, a) \geq -\left(\frac{d-2}{2}\right)^2 + 1$ and, consequently, admits a unique self-adjoint extension (which coincides with the Friedrichs' extension); otherwise, i.e. if $\mu_0(\mathbf{0}, a) < -\left(\frac{d-2}{2}\right)^2 + 1$, H is not essentially self-adjoint and admits infinitely many self-adjoint extensions, among which the Friedrichs' extension is

the only one whose domain is included in the domain of the associated quadratic form (see also [12, Remark 2.5]).

The Friedrichs' extension H naturally extends to a self adjoint operator on the dual \mathcal{D}^* of \mathcal{D} and the unitary group e^{-itH} extends to a group of isometries on the dual of \mathcal{D} which will be still denoted as e^{-itH} (see [7], Section 1.6 for further details). Then for every $\psi_0 \in L^2(\mathbb{R}^d)$,

$$\psi(t, x) := e^{-itH} \psi_0(x) \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}; \mathcal{D}^*),$$

is the unique solution to (26).

Now, by means of (29) and (30) define the following kernel:

$$K(x, y) = \sum_{k=-\infty}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|) \varphi_k\left(\frac{x}{|x|}\right) \overline{\varphi_k\left(\frac{y}{|y|}\right)}, \quad (33)$$

where

$$j_\nu(r) := r^{-\frac{d-2}{2}} J_{\nu+\frac{d-2}{2}}(r)$$

and J_ν denotes the usual Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}.$$

Notice that (33) reduces to (10), in the free case $\mathbf{A} \equiv a \equiv 0$. The first result we mention in this survey is the following representation formula for e^{-itH} :

Theorem 1 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [13]). *Let $d \geq 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^N)$, and assume (27) and (31). Then, for any $\psi_0 \in L^2(\mathbb{R}^d)$,*

$$e^{-itH} \psi_0(x) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{d/2}} \int_{\mathbb{R}^d} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} \psi_0(y) dy. \quad (34)$$

As an immediate consequence, we see by (34) that the analog to condition (7) gives for H the complete list of usual time decay estimates (5):

Corollary 1. *Let $d \geq 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^N)$, and assume (27) and (31). If*

$$\sup_{x, y \in \mathbb{R}^d} |K(x, y)| < \infty, \quad (35)$$

then

$$\|e^{-itH} \psi_0(\cdot)\|_{L^p(\mathbb{R}^d)} \leq C|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|\psi_0\|_{L^{p'}(\mathbb{R}^d)}, \quad \forall t \in \mathbb{R}, \quad \forall p \geq 2, \quad (36)$$

for some $C > 0$ independent on ψ_0 .

In the two last decades, estimates (36) were intensively studied by several authors. The following is an incomplete list of results about this topic [2, 8, 9, 10, 11, 18, 19,

26, 28, 29, 33, 34, 36, 37, 38, 39]. In all these papers, the potentials are sub-critical with respect to the functional scale of the Hardy's inequality (11): in other words, the critical potentials in (25) are never considered, and it does not seem that one could handle them by perturbation techniques, which are a common factor of all the above mentioned papers. Now, formula (34) and Corollary 1 give a usual tool to reduce matters to prove time decay, to a spectral analysis problem. This allowed us to prove some new positive results concerning with estimates (36). In 2D, the picture is quite well understood, thanks to the following theorem.

Theorem 2 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [14]). *Let $d = 2$, $a \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R})$, $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$ satisfying (27) and $\mu_1(\mathbf{A}, a) > 0$, and H be given by (25). Then, for any $\psi_0 \in L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)$,*

$$\|e^{-itH}\psi_0(\cdot)\|_{L^p(\mathbb{R}^2)} \leq C|t|^{-2(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^{p'}(\mathbb{R}^2)}, \quad \forall t \in \mathbb{R}, \quad \forall p \geq 2, \quad (37)$$

for some $C > 0$ independent on ψ_0 .

Theorem 2 is proved in [14]. The core consists in proving that (35) holds, and a crucial role is played by the following Lemma, which gives a quite explicit expansion of eigenvalues and eigenfunctions of L , generalizing the results in [21]:

Lemma 1 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [14]). *Let $a \in W^{1,\infty}(\mathbb{S}^1)$, $\tilde{a} := \frac{1}{2\pi} \int_0^{2\pi} a(s) ds$, $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1)$ such that*

$$\tilde{\mathbf{A}} = \frac{1}{2\pi} \int_0^{2\pi} A(s) ds \notin \frac{1}{2}\mathbb{Z}. \quad (38)$$

Then there exist $k^, \ell \in \mathbb{N}$ such that $\{\mu_k : k > k^*\} = \{\lambda_j : j \in \mathbb{Z}, |j| \geq \ell\}$,*

$$\sqrt{\lambda_j - \tilde{a}} = (\operatorname{sgn} j) \left(\tilde{\mathbf{A}} - \left\lfloor \tilde{\mathbf{A}} + \frac{1}{2} \right\rfloor \right) + |j| + O\left(\frac{1}{|j|^3}\right), \quad \text{as } |j| \rightarrow +\infty$$

and

$$\lambda_j = \tilde{a} + \left(j + \tilde{\mathbf{A}} - \left\lfloor \tilde{\mathbf{A}} + \frac{1}{2} \right\rfloor \right)^2 + O\left(\frac{1}{j^2}\right), \quad \text{as } |j| \rightarrow +\infty. \quad (39)$$

Furthermore, for all $j \in \mathbb{Z}$, $|j| \geq \ell$, there exists a $L^2(\mathbb{S}^1, \mathbb{C})$ -normalized eigenfunction φ_j of the operator L on \mathbb{S}^1 corresponding to the eigenvalue λ_j such that

$$\varphi_j(\theta) = \frac{1}{\sqrt{2\pi}} e^{-i([\tilde{\mathbf{A}}+1/2]\theta + \int_0^\theta A(t) dt)} \left(e^{i(\tilde{\mathbf{A}}+j)\theta} + R_j(\theta) \right), \quad (40)$$

where $\|R_j\|_{L^\infty(\mathbb{S}^1)} = O\left(\frac{1}{|j|^3}\right)$ as $|j| \rightarrow \infty$. In the above formula $\lfloor \cdot \rfloor$ denotes the floor function $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.

Analogous results to Lemma 1 can be proved (and are in part available) in higher dimension $d \geq 3$. Nevertheless, the higher dimensional scenario is quite more complicate, and some chaotic behavior of the eigenvalues of L can occur. This makes the generic validity of (36) completely unclear in dimension $d \geq 3$. In this direction,

the only result which is available at the moment is concerned with the 3D-inverse square electric potential, and reads as follows:

Theorem 3 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [13]). *Let $d = 3$, $\mathbf{A} \equiv 0$ and $a(\omega) \equiv a \in \mathbb{R}$, with $a > -\frac{1}{4}$.*

i) *If $a \geq 0$, then, for any $\psi_0 \in L^2(\mathbb{R}^3) \cap L^{p'}(\mathbb{R}^3)$,*

$$\|e^{-itH}\psi_0(\cdot)\|_{L^p(\mathbb{R}^2)} \leq C|t|^{-3\left(\frac{1}{2}-\frac{1}{p}\right)}\|\psi_0\|_{L^{p'}(\mathbb{R}^2)}, \quad \forall t \in \mathbb{R}, \quad \forall p \geq 2, \quad (41)$$

for some $C > 0$ which does not depend on ψ_0 .

ii) *If $-\frac{1}{4} < a < 0$, let α_1 as in (30), and define*

$$\|\psi\|_{p,\alpha_1} := \left(\int_{\mathbb{R}^3} (1+|x|^{-\alpha_1})^{2-p} |\psi(x)|^p dx \right)^{1/p}, \quad p \geq 1.$$

Then the following estimates hold

$$\|e^{-itH}\psi_0(\cdot)\|_{p,\alpha_1} \leq \frac{C(1+|t|^{\alpha_0})^{1-\frac{2}{p}}}{|t|^{3\left(\frac{1}{2}-\frac{1}{p}\right)}}\|\psi_0\|_{p',\alpha_0}, \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad (42)$$

for some constant $C > 0$ which does not depend on ψ_0 .

Remark 1. It is interesting to remark that, in the range $-1/4 < a < 0$, (41) does not hold, while the full set of usual Strichartz estimates hold (see [5, 6]). This is now clearly understood in terms of formula (34): notice that, if $a = \mu_0 < 0$, then $\alpha_0 > 0$ and a negative-index Bessel function appears in the kernel K given by (33); since negative-index functions J_ν are singular at the origin, one cannot expect the solution (34) to be in L^∞ .

This can be proved as a general fact:

Theorem 4 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [14]). *Let $d \geq 3$, $a \in L^\infty(\mathbb{S}^{d-1}, \mathbb{R})$, $\mathbf{A} \in C^1(\mathbb{S}^{d-1}, \mathbb{R}^d)$, and assume (27), (31), and $\mu_0 < 0$. Then, for almost every $t \in \mathbb{R}$, $e^{-itH}(L^1) \not\subseteq L^\infty$; in particular e^{-itH} is not a bounded operator from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.*

The above phenomenon can be quantified. To this aim, let us restrict our attention to the case

$$H = -\Delta + \frac{a}{|x|^2}, \quad x \in \mathbb{R}^3.$$

Let us define

$$V_{n,j}(x) = |x|^{-\alpha_j} e^{-\frac{|x|^2}{4}} P_{j,n}\left(\frac{|x|^2}{2}\right) \psi_j\left(\frac{x}{|x|}\right), \quad n, j \in \mathbb{N}, \quad j \geq 1, \quad (43)$$

where $P_{j,n}$ is the polynomial of degree n given by

$$P_{j,n}(t) = \sum_{i=0}^n \frac{(-n)_i}{\left(\frac{d}{2} - \alpha_j\right)_i} \frac{t^i}{i!},$$

denoting as $(s)_i$, for all $s \in \mathbb{R}$, the Pochhammer's symbol

$$(s)_i = \prod_{j=0}^{i-1} (s+j), \quad (s)_0 = 1.$$

Moreover, for all $k > 1$, define

$$\mathcal{U}_k = \text{span} \{V_{n,j} : n \in \mathbb{N}, 1 \leq j < k\} \subset L^2(\mathbb{R}^N).$$

The functions $V_{n,j}$ spans $L^2(\mathbb{R}^3)$ (see [15] for details). Moreover, as initial data for (1), these functions have a quite explicit evolution: indeed, denoting by $\tilde{V}_{n,j} := V_{n,j}/\|V_{n,j}\|_2$, the following identity holds:

$$\begin{aligned} e^{-itH} \tilde{V}_{n,j}(x) &= e^{it\left(-\Delta + \frac{a}{|x|^2}\right)} V_{n,j}(x) \\ &= (1+t^2)^{-\frac{d}{4} + \frac{\alpha_j}{2}} |x|^{-\alpha_j} \frac{e^{\frac{-|x|^2}{4(1+t^2)}}}{\|V_{n,j}\|_{L^2(\mathbb{R}^d)}} e^{i\frac{|x|^2 t}{4(1+t^2)}} e^{-i\gamma_{n,j} \arctan t} \psi_j\left(\frac{x}{|x|}\right) P_{n,j}\left(\frac{|x|^2}{2(1+t^2)}\right). \end{aligned} \quad (44)$$

Formula (44) has been proved in [15]. Clearly, if $a = \mu_0 \geq 0$, then $\alpha_0 \leq 0$ and the first function $\tilde{V}_{1,0}$ decays polynomially faster than usual, in a weighted space. This is reminiscent to question **B** in the Introduction, and gives us the following evolution version of the frequency-dependent Hardy's inequality (19):

Theorem 5 (L. Fanelli, V. Felli, M. Fontelos, A. Primo - [15]). *Let $d = 3$, $a = \mu_0 \geq 0$, α_0 as in (30).*

(i) *There exists $C > 0$ such that, for all $\psi_0 \in L^2(\mathbb{R}^3)$ with $|x|^{-\alpha_0} \psi_0 \in L^1(\mathbb{R}^3)$,*

$$\left\| |x|^{\alpha_0} e^{-itH} \psi_0(\cdot) \right\|_{L^\infty} \leq C |t|^{-\frac{3}{2} + \alpha_0} \left\| |x|^{-\alpha_0} \psi_0 \right\|_{L^1}.$$

(ii) *For all $k \in \mathbb{N}$, $k \geq 1$, there exists $C_k > 0$ such that, for all $\psi_0 \in \mathcal{U}_k^\perp$ with $|x|^{-\alpha_k} \psi_0 \in L^1(\mathbb{R}^3)$,*

$$\left\| |x|^{\alpha_k} e^{-itH} \psi_0(\cdot) \right\|_{L^\infty} \leq C_k |t|^{-\frac{3}{2} + \alpha_k} \left\| |x|^{-\alpha_k} \psi_0 \right\|_{L^1}.$$

Some analogous results, only concerning with the decay of the first frequency space, had been previously proven in [16, 20].

To complete the survey, we leave some open questions.

- (i) Concerning Theorems 2, 3, does any general result hold in dimension $d \geq 3$?
- (ii) In what extent can one perturb the models in (25)? What is the real role played by the scaling invariance?

- (iii) The proof of formula (34) strongly relies on some pseudoconformal law associated to the free Schrödinger flow (Appell transform; see [13]). Is there any analog for other dispersive models, e.g. the wave equation?
- (iv) One can use formula (34) to represent the wave operators

$$W_{\pm} := L^2 - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}, \quad H_0 := -\Delta.$$

What can one prove about the boundedness of W_{\pm} in $L^p(\mathbb{R}^d)$, in the same style as in [33, 34, 36, 37, 38, 39] (at least in 2D, having in mind Theorem 2.

- (v) By standard TT^* -arguments, one can obtain some weighted Strichartz estimates by Theorem 5. Which kind of informations do these estimates give for nonlinear Schrödinger equations associated to H ?

Acknowledgements The author sincerely thanks all the organizers of the event *Contemporary Trends in the Mathematics of Quantum Mechanics* for the kind invitation, in particular Alessandro Michelangeli for his job. A special acknowledgement is to Prof. Gianfausto Dell’Antonio, for his beautiful scientific contributions, for his teachings, and for his friendship.

References

1. M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series **55**. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964.
2. M. BECEANU AND M. GOLDBERG, Decay estimates for the Schrödinger equation with critical potentials, to appear in *Comm. Math. Phys.*, arXiv:1009.5285.
3. A. BEZUBIK AND A. STRASBURGER, A new form of the spherical expansion of zonal functions and Fourier transforms of $SO(d)$ -finite functions, *SIGMA Symmetry Integrability Geom. Methods Appl.* **2** (2006), Paper 033, 8 pp.
4. G. BORG, Umkehrung der Sturm-Liouvillischen Eigenwertanfrage Bestimmung der Differentialgleichung die Eigenwerte, *Acta Math.* **78** (1946), 1-96.
5. BURQ, N., PLANCHON, F., STALKER, J., AND TAHVILDAR-ZADEH, S., Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, *J. Funct. Anal.* **203** (2003) no. 2, 519–549.
6. N. BURQ, F. PLANCHON, J. STALKER, AND S. TAHVILDAR-ZADEH, Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, *Indiana Univ. Math. J.* **53**(6) (2004), 1665–1680.
7. T. CAZENAVE, *Semilinear Schrödinger equations*. Courant Lecture Notes in Mathematics **10**, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.
8. ERDOGAN, M.B., GOLDBERG, M., AND SCHLAG, W., Strichartz and Smoothing Estimates for Schrödinger Operators with Almost Critical Magnetic Potentials in Three and Higher Dimensions, *Forum Math.* **21** (2009), 687–722.
9. M.B. ERDOGAN, M. GOLDBERG, AND W. SCHLAG, Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in \mathbb{R}^3 , *J. European Math. Soc.* **10** (2008), 507–531.
10. P. D’ANCONA AND L. FANELLI, L^p -boundedness of the wave operator for the one dimensional Schrödinger operators, *Comm. Math. Phys.* **268** (2006), 415–438.

11. P. D'ANCONA AND L. FANELLI, Decay estimates for the wave and Dirac equations with a magnetic potential, *Comm. Pure Appl. Math.* **60** (2007), 357–392.
12. T. DUYCKAERTS, Inégalités de résolvante pour l'opérateur de Schrödinger avec potentiel multipolaire critique, *Bulletin de la Société mathématique de France* **134** (2006), 201–239.
13. L. FANELLI, V. FELLI, M. FONTELOS, AND A. PRIMO, Time decay of scaling critical electromagnetic Schrödinger flows, *Communications in Mathematical Physics* **324** (2013), 1033–1067.
14. L. FANELLI, V. FELLI, M. FONTELOS, AND A. PRIMO, Time decay of scaling invariant electromagnetic Schrödinger equations on the plane, *Communications in Mathematical Physics* **337** (2015), 1515–1533.
15. L. FANELLI, V. FELLI, M. FONTELOS, AND A. PRIMO, Frequency-dependent time decay of Schrödinger flows, to appear in *J. Spectral Theory*.
16. FANELLI, L., GRILLO, G., AND KOVAŘÍK, H., Improved time-decay for a class of scaling critical electromagnetic Schrödinger flows, *J. Func. Anal.* **269** (2015), 3336–3346.
17. V. FELLI, A. FERRERO, AND S. TERRACINI, Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential, *J. Eur. Math. Soc.* **13** (2011) no. 1, 119–174.
18. M. GOLDBERG, Dispersive estimates for the three-dimensional schrödinger equation with rough potential, *Amer. J. Math.* **128** (2006), 731–750.
19. M. GOLDBERG AND W. SCHLAG, Dispersive estimates for Schrödinger operators in dimensions one and three, *Comm. Math. Phys.* **251** (2004) no. 1, 157–178.
20. G. GRILLO AND H. KOVARIK, Weighted dispersive estimates for two-dimensional Schrödinger operators with Aharonov-Bohm magnetic field, *Journal of Differential Equations* **256** (2014), 3889–3911.
21. D. GURARIE, Zonal Schrödinger operators on the n -Sphere: Inverse Spectral Problem and Rigidity, *Comm. Math. Phys.* **131** (1990), 571–603.
22. M. E. H. ISMAIL, *Classical and quantum orthogonal polynomials in one variable*. Encyclopedia of Mathematics and its Applications, 98. Cambridge University Press, Cambridge, 2005.
23. H. KALF, U.-W. SCHMINCKE, J. WALTER, R. WÜST, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, Spectral theory and differential equations (Proc. Sympos., Dundee, 1974; dedicated to Konrad Jörgens), pp. 182–226. Lecture Notes in Math., Vol. 448, Springer, Berlin, 1975.
24. M. KEEL AND T. TAO, Endpoint Strichartz estimates, *Am. J. Math.* **120** no. 5 (1998), 955–980.
25. LAPTEV, A. AND WEIDL, T., Hardy inequalities for magnetic Dirichlet forms, *Mathematical results in quantum mechanics (Prague, 1998)*, 299–305; *Oper. Theory Adv. Appl.* **108**, Birkhäuser, Basel, 1999.
26. F. PLANCHON, J. STALKER, AND S. TAHVILDAR-ZADEH, Dispersive estimates for the wave equation with the inverse-square potential, *Discrete Contin. Dyn. Syst.* **9** (2003), 1387–1400.
27. M. REED AND B. SIMON, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York-London, 1975.
28. I. RODNIANSKI AND W. SCHLAG, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, *Invent. Math.* **155** (2004) no. 3, 451–513.
29. W. SCHLAG, Dispersive estimates for Schrödinger operators: a survey, *Mathematical aspects of nonlinear dispersive equations*, 255285, *Ann. of Math. Stud.*, **163**, Princeton Univ. Press, Princeton, NJ, 2007.
30. B. SIMON, Essential self-adjointness of Schrödinger operators with singular potentials, *Arch. Rational Mech. Anal.* **52** (1973), 44–48.
31. L.E. THOMAS AND C. VILLEGAS-BLAS, Singular Continuous Limiting Eigenvalue Distributions for Schrödinger operators on a 2-Sphere, *J. Func. Anal.* **141** (1996), 249–273.
32. L. E. THOMAS AND S. R. WASSELL, Semiclassical Approximation for Schrödinger operators on a two-sphere at high energy, *J. Math. Phys.* **36** (1995) no. 10, 5480–5505.
33. R. WEDER, The $W_{k,p}$ -continuity of the Schrödinger Wave Operators on the line, *Comm. Math. Phys.* **208** (1999), 507–520.

- 34. R. WEDER, $L^p - L^{p'}$ estimates for the Schrödinger equations on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, *J. Funct. Anal.* **170** (2000), 37–68.
- 35. A. WEINSTEIN, Asymptotics for eigenvalue clusters for the laplacian plus a potencial, *Duke Math. J.* **44** (1977), no. 4, 883..892.
- 36. K. YAJIMA, Existence of solutions for Schrödinger evolution equations, *Comm. Math. Phys.* **110** (1987), 415–426.
- 37. K. YAJIMA, The $W^{k,p}$ -continuity of wave operators for Schrödinger operators, *J. Math. Soc. Japan* **47** (1995) no. 3, 551–581.
- 38. K. YAJIMA, The $W^{k,p}$ -continuity of wave operators for Schrödinger operators III, even dimensional cases $m \geq 4$, *J. Math. Sci. Univ. Tokyo* **2** (1995), 311–346.
- 39. K. YAJIMA, L^p -boundedness of wave operators for two-dimensional Schrödinger operators, *Comm. Math. Phys.* **208** (1999) no. 1, 125–152.